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ORIGINAL RESEARCH

Free vibration analysis of axisymmetric laminated composite circular and annular plates using Chebyshev collocation

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Abstract Solutions, based on principle of collocating the equations of motion at Chebyshev zeroes, are presented for the free vibration analysis of laminated, polar orthotropic, circular and annular plates. The analysis is restricted to axisymmetric free vibration of the plates and employs first-order shear deformation theory for the displacement field, in terms of midplane displacements, u, α and w. The eigenvalue problem is defined in terms of three equations of motion in terms of the radial co-ordinate r, the radial variation of the displacements being represented in polynomial series, with appropriate boundary conditions. Numerical results are presented to show the validity and accuracy of the proposed method. Results of parametric studies for laminated polar orthotropic circular and annular plates with different boundary conditions, orthotropic ratios, lamination sequences, number of layers and shear deformation are also presented.

Keywords Free vibration · Chebyshev collocation method · First-order shear deformation theory · Laminated composites

List of symbols

r, θ, z	Cylindrical co-ordinates
t	Time
U	Inplane displacement in r direction
V	Inplane displacement in θ direction
W	Inplane displacement in z direction
α_1	Rotation in r direction
α_2	Rotation in θ direction
σ_r, σ_θ, σ_z	Normal stresses along principal material directions
$\tau_{r\theta}$, $\tau_{\theta z}$, τ_{rz}	Shear stresses along principal material directions
ε_r, ε_θ, ε_z	Normal strains along principal material directions
$\gamma_{r\theta}$, $\gamma_{\theta z}$, γ_{rz}	Shearing strains along principal material directions
ε_r^0, ε_θ^0, $\gamma_{r\theta}^0$	Midplane strains of a laminate
κ_r, κ_θ, $\kappa_{r\theta}$	Midplane curvatures of a laminate
C_{ij}	Plane-stress reduced lamina stiffnesses
a, b	Outer and inner radii of the laminated plate
h	Total thickness of the laminate
Z	Distance of lamina from midplane
N_r, N_θ, $N_{r\theta}$	Inplane mechanical stress resultants
M_r, M_θ, $M_{r\theta}$	Moment mechanical stress resultants
Q_r, Q_θ	Transverse shear stress resultants
A_{ij}	Extension stiffness
B_{ij}	Bending–extension coupling stiffness
D_{ij}	Bending stiffness
N	Number of collocation points
K^2	Shear correction factor
E_r, E_θ	Modulus of elasticity
E, υ	Young's modulus and Poisson's ratio of an isotropic material
D	Bending stiffness of an isotropic material
$G_{r\theta}$, G_{rz}	Shear modulus
$\upsilon_{r\theta}$	Poisson's ratio of the plate material
n	Number of layers

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ξ	Nondimensional radial co-ordinate
U, W, α	Nondimensional displacement components
q	Transverse load intensity
p	Nondimensional load intensity
ρ	Density
λ	Nondimensional frequency parameter
ω	Axisymmetric frequency parameter
C	Clamped boundary condition
S	Simply supported boundary condition
$[K]$, $[M]$	Stiffness and mass matrices of the laminate
$T_r(x)$	rth-order Chebyshev polynomial
$T_r^*(\xi)$	The shifted Chebyshev polynomial in the specified range

Introduction

Fiber-reinforced composite structures are often subjected to dynamic loading caused by time-dependent loads causing vibrations or wave propagation. Then, the response of these structures under time-varying loads depends not only on the distribution of the stiffness of material in the structure, but also on the distribution of mass inertia. The analyst has to then consider the effect of inertia forces set up within the structure at any instant. The studies of flexural vibrations of plates subjected to different boundary conditions have thus received considerable interest because of their technological importance, and also give a good idea of response characteristics of the structure under dynamic loads.

Circular and annular plates are commonly used structural components in aerospace, civil, mechanical, electronic and nuclear engineering applications. In industrial situations, it is often required to predict the free vibration characteristics of these plates. For the free vibration analysis of various plates, there are a number of solution techniques, such as analytical methods, energy methods, finite difference methods and finite element methods. Analytical solutions form an important basis for comparison and verification of results obtained by numerical methods such as the finite element method. Among the different m is also the Chebyshev collocation method.

There have been a number of investigations of the free vibration of homogeneous isotropic circular and annular plates such as Han and Liew (1999), Haterbouch and Benamar (2005), Liew et al. (1997), Liew and Yang (1999), Liew and Yang (2000), Selmane and Lakis (1999) and Zhou et al. (2003). There are works employing solutions using differential quadrature and generalized differential quadrature methods for the study of this class of problems and so also a few finite element solutions for the analysis of laminated plates and shells such as Han and Liew (1997), Lin and Tseng (1998), Ding and Xu (2000), Liew and Liu (2000), Wu et al. (2002), Tornabene et al. (2009) and Hosseini-Hashemi et al. (2010). In a recent paper (Xiang et al. 2014), the equations of motion of composite laminated annular plates, conical and cylindrical shells, with various boundary conditions based on the first-order shear deformation theory, have been solved for natural frequencies using an innovative, Haar wavelet discretization method. However, there are not many studies showing use of collocation at Chebyshev zeroes as an effective solution methodology for the determination of natural frequencies of laminated circular and annular plates.

In the present work, it is proposed to study the free vibration characteristics of laminated polar orthotropic circular and annular plates by Chebyshev collocation method. The possible application of orthogonal collocation to boundary value problems has been discussed by Villadsen and Stewart as early as (1967). Carey and Finlayson (1974) have explored the concept of orthogonal collocation in finite element analysis. The method has been used earlier for solving problems of free vibration analysis and large amplitude deflection analysis of isotropic and orthotropic spherical shells—static analysis (Dumir et al. 1984; Nath and Jain 1986). Dumir et al. (2001) have presented geometrically nonlinear analysis of a moderately thick, laminated composite annular plate subjected to uniformly distributed ring loads. Narasimhan (1992) has analyzed the problem of dynamic response analysis of laminated spherical shells using the same method. Herein, the possible application of the methodology for solution of axisymmetric free vibration response of circular and annular (polar) orthotropic plates has been illustrated.

In the present research, the reference plane displacements u, α and w are expanded in polynomial series and then orthogonal point collocation method is used to discretise the governing equations. The eigenvalue problem is derived from the equations of motion, neglecting the rotary inertia and inplane inertia terms. To demonstrate the convergence of the method, numerical results are presented for clamped and simply supported isotropic and polar orthotropic circular and annular plates. The validity of the solution methodology adopted is confirmed by comparing nondimensional frequencies for isotropic and polar orthotropic plates obtained from the proposed solution with data obtained from open literature. It is observed that the present method is efficient in obtaining the free vibration frequencies and mode shapes of the laminated circular and annular plates made of composite materials. Parametric studies are also conducted and it is concluded that free vibration frequencies are dependent not only on the boundary conditions, but also on the parameters such as fiber orientation, lamination sequence and hole diameter.

Methods

Mathematical formulation

The laminated plate of constant thickness h is composed of polar orthotropic laminae stacked in any arbitrary sequence, but with their fiber reinforcement aligned either in radial or circumferential directions only is considered. Polar co-ordinates (r, θ, z) are used for plate co-ordinates as shown in Fig. 1, where u, v, w denote the displacements of any point of the plate in the corresponding r, θ, z directions.

First-order shear deformation theory is employed in the present study and the displacement field is assumed to be of the form

$$
\begin{aligned}
u(r,\theta,z) &= u^0(r,\theta) + z\alpha_1(r,\theta),\\
v(r,\theta,z) &= v^0(r,\theta) + z\alpha_2(r,\theta),\\
w(r,\theta,z) &= w^0(r,\theta)
\end{aligned} \tag{1}
$$

where u^0, v^0, w^0 denote the displacements of any point on the middle surface and α_1, α_2 are the rotations of the normal to the midplane about θ, r axes, respectively.

The linear strain displacement relations for the general motion of a point on the reference surface of laminated orthotropic circular plates are given by

$$
\begin{aligned}
&\varepsilon_r = \varepsilon_r^0 + z\cdot\kappa_r,\quad \varepsilon_\theta = \varepsilon_\theta^0 + z\cdot\kappa_\theta,\quad \gamma_{r\theta} = \gamma_{r\theta}^0 + z\cdot\kappa_{r\theta},\\
&\gamma_{rz} = \gamma_{rz}^0,\quad \gamma_{\theta z} = \gamma_{\theta z}^0
\end{aligned} \tag{2}
$$

where the reference surface strains and curvatures are given by

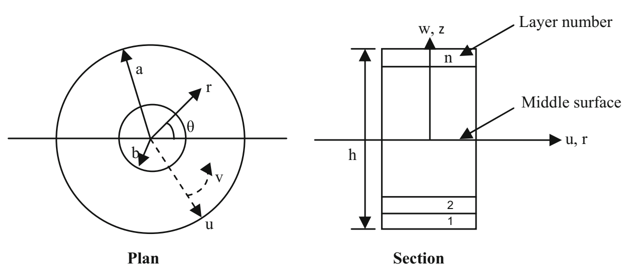

Fig. 1 Geometry of n-layered laminate

$$
\begin{aligned}
&\varepsilon_r^0 = \frac{\partial u^0}{\partial r},\quad \varepsilon_\theta^0 = \frac{1}{r}\cdot\left(\frac{\partial v^0}{\partial\theta} + u^0\right)\\
&\gamma_{r\theta}^0 = \frac{1}{r}\cdot\left(\frac{\partial u^0}{\partial\theta} - v^0\right) + \frac{\partial v^0}{\partial r},\\
&\gamma_{rz}^0 = \alpha_1 + \frac{\partial w}{\partial r},\quad \gamma_{\theta z}^0 = \alpha_2 + \frac{1}{r}\cdot\left(\frac{\partial w}{\partial\theta}\right)\\
&\kappa_r = \frac{\partial\alpha_1}{\partial r},\quad \kappa_\theta = \frac{1}{r}\cdot\left(\frac{\partial\alpha_2}{\partial\theta} + \alpha_1\right),\\
&\kappa_{r\theta} = \frac{1}{r}\cdot\left(\frac{\partial\alpha_1}{\partial\theta} - \alpha_2\right) + \frac{\partial\alpha_2}{\partial r}
\end{aligned} \tag{3}
$$

According to the shear deformation theory, the constitutive equations for the kth layer of a polar orthotropic laminated plate can be written in the following form in polar co-ordinates

$$
\begin{aligned}
&\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{Bmatrix}^{(k)} = \begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix}^{(k)} \begin{Bmatrix}\varepsilon_r\\ \varepsilon_\theta\\ \gamma_{r\theta}\end{Bmatrix}^{(k)},\\
&\begin{Bmatrix}\tau_{rz}\\ \tau_{\theta z}\end{Bmatrix}^{(k)} = \begin{bmatrix}C_{44} & 0\\ 0 & C_{55}\end{bmatrix}^{(k)} \begin{Bmatrix}\gamma_{rz}\\ \gamma_{\theta z}\end{Bmatrix}^{(k)},
\end{aligned} \tag{4}
$$

where the elastic constants are expressed in terms of material constants of the lamina in the plate co-ordinates as

$$
\begin{aligned}
&C_{11} = \frac{E_r}{1-\upsilon_{r\theta}\upsilon_{\theta r}}\quad C_{12} = \frac{\upsilon_{r\theta}E_\theta}{1-\upsilon_{r\theta}\upsilon_{\theta r}} = \frac{\upsilon_{\theta r}E_r}{1-\upsilon_{r\theta}\upsilon_{\theta r}}\\
&C_{22} = \frac{E_\theta}{1-\upsilon_{r\theta}\upsilon_{\theta r}}\\
&C_{44} = G_{rz},\quad C_{55} = G_{\theta z},\quad C_{66} = G_{r\theta}
\end{aligned} \tag{5}
$$

where E_r and E_θ are Young's moduli of elasticity in r and θ directions. $\upsilon_{r\theta}$ and $\upsilon_{\theta r}$ are Poisson's ratios. $G_{r\theta}$, $G_{\theta z}$ and G_{rz} are the shear moduli in the respective planes.

The stress resultants acting on a laminate are obtained as:

$$
\begin{aligned}
\begin{bmatrix}N_r & M_r\\ N_\theta & M_\theta\\ N_{r\theta} & M_{r\theta}\end{bmatrix} &= \int_{-h/2}^{h/2} \begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{Bmatrix}(1,z)\mathrm{d}z\\
&= \sum_{k=1}^{n}\int_{z_{k-1}}^{z_k} \begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{Bmatrix}^{(k)} (1z)\,\mathrm{d}z,\\
\begin{Bmatrix}Q_r\\ Q_\theta\end{Bmatrix} = K^2\int_{-h/2}^{h/2}\begin{Bmatrix}\tau_{rz}\\ \tau_{\theta z}\end{Bmatrix}\mathrm{d}z &= K^2\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k}\begin{Bmatrix}\tau_{rz}\\ \tau_{\theta z}\end{Bmatrix}^{(k)}\mathrm{d}z
\end{aligned} \tag{6}
$$

where z is the distance of the lamina from the middle plane.

The first-order shear deformation theory used herein assumes a constant state of transverse shear strain through the thickness of the plate and hence requires shear correction factors introduced to account for non-uniform distribution of the transverse shear strains through the thickness of the plate. In Eq. (6), K^2 is the Shear correction factor introduced to account for non-uniform distribution of the transverse shear strains through the thickness of the plate, which is taken as $\pi^2/12$.

Substituting the stress strain relations in the expressions for stress resultants, we have

$$\begin{Bmatrix} N_r \\ N_\theta \\ N_{r\theta} \end{Bmatrix} = \sum_{k=1}^{n} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}^{(k)} \left\{ \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \overset{\circ}{\varepsilon}_r \\ \overset{\circ}{\varepsilon}_\theta \\ \overset{\circ}{\gamma}_{r\theta} \end{Bmatrix} \mathrm{d}z + \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_{r\theta} \end{Bmatrix} z\,\mathrm{d}z \right\}$$
$$\begin{Bmatrix} M_r \\ M_\theta \\ M_{r\theta} \end{Bmatrix} = \sum_{k=1}^{n} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}^{(k)} \left\{ \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \overset{\circ}{\varepsilon}_r \\ \overset{\circ}{\varepsilon}_\theta \\ \overset{\circ}{\gamma}_{r\theta} \end{Bmatrix} z\,\mathrm{d}z + \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_{r\theta} \end{Bmatrix} z^2\mathrm{d}z \right\} \tag{7}$$
$$\begin{Bmatrix} Q_r \\ Q_\theta \end{Bmatrix} = K^2 \sum_{k=1}^{n} \begin{bmatrix} C_{44} & 0 \\ 0 & C_{55} \end{bmatrix}^{(k)} \left\{ \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \overset{\circ}{\gamma}_{rz} \\ \overset{\circ}{\gamma}_{\theta z} \end{Bmatrix} \mathrm{d}z \right\}.$$

Since $\varepsilon_r^0, \varepsilon_\theta^0, \gamma_{r\theta}^0, \kappa_r, \kappa_\theta, \kappa_{r\theta}, \gamma_{rz}^0, \gamma_{\theta z}^0$ are middle surface strains and curvatures and not functions of z, they can be taken out of the integration signs. Thus, the total plate constitutive equations can be written as

$$\begin{Bmatrix} N_r \\ N_\theta \\ N_{r\theta} \\ M_r \\ M_\theta \\ M_{r\theta} \\ Q_r \\ Q_\theta \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} & 0 & 0 \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 & 0 & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 & 0 & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K^2A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K^2A_{55} \end{bmatrix} \begin{Bmatrix} \overset{\circ}{\varepsilon}_r \\ \overset{\circ}{\varepsilon}_\theta \\ \overset{\circ}{\gamma}_{r\theta} \\ \kappa_r \\ \kappa_\theta \\ \kappa_{r\theta} \\ \overset{\circ}{\gamma}_{rz} \\ \overset{\circ}{\gamma}_{\theta z} \end{Bmatrix} \tag{8}$$

where, $A_{ij} = \sum_{k=1}^{n} (C_{ij})^{(k)}(z_k - z_{k-1}), \quad i,j = 1,2,6,4,5$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n} (C_{ij})^{(k)}(z_k^2 - z_{k-1}^2), \quad i,j = 1,2,6$$
$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} (C_{ij})^{(k)}(z_k^3 - z_{k-1}^3), \quad i,j = 1,2,6 \tag{9}$$

A_{ij} is the extensional stiffness, B_{ij} is the bending–extension coupling stiffness, D_{ij} is the bending stiffness.

If the plates are subjected to transverse loads only, the stress resultants and stress couples must satisfy the following equilibrium equations (Ravichandran 1989)

$$\frac{\partial N_r}{\partial r}+\frac{1}{r}\frac{\partial N_{r\theta}}{\partial\theta}+\frac{(N_r-N_\theta)}{r}=0,$$
$$\frac{\partial Q_r}{\partial r}+\frac{1}{r}\frac{\partial Q_\theta}{\partial\theta}+\frac{Q_r}{r}=I_o\frac{\partial^2 w}{\partial t^2},\frac{\partial M_r}{\partial r}+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}+\frac{(M_r-M_\theta)}{r}=Q_r$$
$$\frac{\partial N_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial N_\theta}{\partial\theta}+2\frac{N_{r\theta}}{r}=0,\quad \frac{\partial M_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial M_\theta}{\partial\theta}+2\frac{M_{r\theta}}{r}=Q_\theta \tag{10}$$

where
$$I_0=\sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\rho^{(k)}\,dz=\rho h\sum_{k=1}^{n}\left\{\frac{\rho^{(k)}}{\rho h}(h_k-h_{k-1})\right\}=\rho\,hR_m \tag{11}$$

where
$$R_m=\sum_{k=1}^{n}\left\{\frac{\rho^{(k)}}{\rho\,h}(h_k-h_{k-1})\right\} \tag{11a}$$

and ρ is a reference density.

For axisymmetric case, the stresses and strains are independent of θ and $\tau_{r\theta}=\tau_{z\theta}=0$, $v^0=0$, $\alpha_2=0$ and also $\frac{\partial}{\partial\theta}(\ )=0$. This also leads to $N_{r\theta}=0$, $M_{r\theta}=0$ and $Q_\theta=0$.

Substituting for stress resultants and stress couples in Eq. (10) in terms of strains and curvatures which in turn are substituted in terms of displacements given by Eq. (3), the equations take the form

$$A_{11}\left(\frac{\partial^2u^0}{\partial r^2}+\frac{\partial u^0}{r\cdot\partial r}\right)-A_{22}\frac{u^0}{r^2}+B_{11}\left(\frac{\partial^2\alpha_1}{\partial r^2}+\frac{\partial\alpha_1}{r\cdot\partial r}\right)-B_{22}\frac{\alpha_1}{r^2}=0$$
$$B_{11}\left(\frac{\partial^2u^0}{\partial r^2}+\frac{\partial u^0}{r\cdot\partial r}\right)-B_{22}\frac{u^0}{r^2}+D_{11}\left(\frac{\partial^2\alpha_1}{\partial r^2}+\frac{\partial\alpha_1}{r\cdot\partial r}\right)-D_{22}\frac{\alpha_1}{r^2}=K^2A_{44}\left(\alpha_1+\frac{\partial w}{\partial r}\right)$$
$$K^2A_{44}\left(\frac{\partial\alpha_1}{\partial r}+\frac{\alpha_1}{r}+\frac{\partial^2w}{\partial r^2}+\frac{\partial w}{r\cdot\partial r}\right)=I_0\frac{\partial^2w}{\partial t^2}. \tag{12}$$

The following dimensionless parameters are introduced for convenience

$$U=\frac{u^0\cdot(a-b)}{h^2},\quad W=\frac{w^0}{h},\quad \alpha=\frac{\alpha_1\cdot(a-b)}{h},$$
$$\xi=\frac{r-b}{a-b},\quad p=\frac{q\cdot(a-b)^4}{E_T\cdot h^4},$$
$$a_{44}=\frac{K^2\cdot A_{44}}{E_T\cdot h},\quad b_{ij}=\frac{B_{ij}}{E_T\cdot h^2},\quad d_{ij}=\frac{D_{ij}}{E_T\cdot h^3}. \tag{13}$$

Here, E_T is the reference Young's modulus. In case of laminated composites with layers of same material, E_T can be taken conveniently to be the Young's modulus in the direction transverse to fiber direction.

Using the nondimensional quantities defined in (13), a set of equations of motion can now be written as

$$a_{11}\left(\frac{1}{(a-b)^2}\cdot\frac{\partial^2U}{\partial\xi^2}+\frac{1}{(\xi\,a+(1-\xi)b)*(a-b)}\cdot\frac{\partial U}{\partial\xi}\right)-a_{22}\frac{U}{(\xi\,a+(1-\xi)b)^2}+b_{11}\left(\frac{1}{(a-b)^2}\cdot\frac{\partial^2\alpha}{\partial\xi^2}+\frac{1}{(\xi\,a+(1-\xi)b)*(a-b)}\cdot\frac{\partial\alpha}{\partial\xi}\right)-b_{22}\frac{\alpha}{(\xi\,a+(1-\xi)b)^2}=0$$
$$b_{11}\left(\frac{1}{(a-b)^2}\cdot\frac{\partial^2U}{\partial\xi^2}+\frac{1}{(\xi\,a+(1-\xi)b)*(a-b)}\cdot\frac{\partial U}{\partial\xi}\right)-b_{22}\frac{U}{(\xi\,a+(1-\xi)b)^2}+d_{11}\left(\frac{1}{(a-b)^2}\cdot\frac{\partial^2\alpha}{\partial\xi^2}+\frac{1}{(\xi\,a+(1-\xi)b)*(a-b)}\cdot\frac{\partial\alpha}{\partial\xi}\right)-d_{22}\frac{\alpha}{(\xi\,a+(1-\xi)b)^2}=\frac{a_{44}}{h^2}\left(\alpha+\frac{\partial W}{\partial\xi}\right)$$
$$a_{44}E_Th^2\left(\frac{1}{(a-b)}\cdot\frac{\partial\alpha}{\partial\xi}+\frac{\alpha}{(\xi\,a+(1-\xi)b)}+\frac{1}{(a-b)}\cdot\frac{\partial^2W}{\partial\xi^2}+\frac{1}{(\xi\,a+(1-\xi)b)}\cdot\frac{\partial W}{\partial\xi}\right)=I_o\frac{\partial^2W}{\partial t^2}. \tag{14}$$

Polynomial series solution by collocation at Chebyshev zeroes

To set up the eigenvalue problem for determination of free vibration frequencies and the corresponding mode shapes, in the present work, Chebyshev collocation method is used. The dependent variables U, α and W and their derivatives are expressed in Chebyshev series as

$$\{U(\xi,t),\alpha(\xi,t),W(\xi,t)\}=\left\{\sum_{n=1}^{N+1}(U_n,\alpha_n,W_n)\xi^{n-1}\right\}e^{i\omega t}. \tag{15}$$

Using Eq. (15) the equations of motion can now be written as

$$\begin{aligned}
&a_{11}\sum_{n=1}^{N+1}\left(\frac{(n-1)(n-2)U_n\xi^{n-3}}{(a-b)^2}+\frac{(n-1)U_n\xi^{n-2}}{(\xi a+(1-\xi)b)*(a-b)}\right)\\
&-a_{22}\sum_{n=1}^{N+1}\frac{U_n\xi^{n-1}}{(\xi a+(1-\xi)b)^2}+\\
&b_{11}\sum_{n=1}^{N+1}\left(\frac{(n-1)(n-2)\alpha_n\xi^{n-3}}{(a-b)^2}+\frac{(n-1)\alpha_n\xi^{n-2}}{(\xi a+(1-\xi)b)*(a-b)}\right)\\
&-b_{22}\sum_{n=1}^{N+1}\frac{\alpha_n\xi^{n-1}}{(\xi a+(1-\xi)b)^2}=0\\
&b_{11}\sum_{n=1}^{N+1}\left(\frac{(n-1)(n-2)U_n\xi^{n-3}}{(a-b)^2}+\frac{(n-1)U_n\xi^{n-2}}{(\xi a+(1-\xi)b)*(a-b)}\right)\\
&-b_{22}\sum_{n=1}^{N+1}\frac{U_n\xi^{n-1}}{(\xi a+(1-\xi)b)^2}+\\
&d_{11}\sum_{n=1}^{N+1}\left(\frac{(n-1)(n-2)\alpha_n\xi^{n-3}}{(a-b)^2}+\frac{(n-1)\alpha_n\xi^{n-2}}{(\xi a+(1-\xi)b)*(a-b)}\right)\\
&-d_{22}\sum_{n=1}^{N+1}\frac{\alpha_n\xi^{n-1}}{(\xi a+(1-\xi)b)^2}
\end{aligned} \tag{16}$$

$$-\frac{a_{44}}{h^2}\sum_{n=1}^{N+1}\left(\alpha_n\xi^{n-1}+(n-1)W_n\xi^{n-2}\right)=0 \tag{17}$$

$$\begin{aligned}
&\frac{a_{44}\cdot a^4}{(a-b)d11\cdot h^2}\sum_{n=1}^{N+1}\left(\frac{(n-1)\,\alpha_n\,\xi^{n-2}}{(a-b)}+.\frac{\alpha_n\xi^{n-1}}{(\xi a+(1-\xi)b)}\right.\\
&\left.+\frac{(n-1)(n-2)\,W_n\xi^{n-3}}{(a-b)}+\frac{(n-1)\,W_n\xi^{n-2}}{(\xi a+(1-\xi)b)}\right)=-\lambda^2\sum_{n=1}^{N+1}W_n\,\xi^{n-1}
\end{aligned} \tag{18}$$

$$\text{where } \lambda=\omega a^2\sqrt{\rho h/D_{11}}. \tag{19}$$

Boundary conditions considered: The following combinations of boundary conditions have been considered in the present work.

Clamped boundary condition

At outer boundary $r=a$, $u^0=\alpha_1^0=w^0=0$ for both circular plate and annular plate.

At inner boundary $r=b$,
$u^0=\alpha_1^0=\mathrm{d}w^0/\mathrm{d}r=0$ for circular plates and
$u^0=\alpha_1^0=w^0=0$ for annular plates.

Simply supported condition type

At outer boundary $r=a$, $u^0=M_r=w^0=0$ for both circular plate and annular plate.

At inner boundary $r=b$,
$u^0=\alpha_1^0=\mathrm{d}w^0/\mathrm{d}r=0$ for circular plates and
$u^0=M_r=w^0=0$ for annular plates.

Simply supported condition type

At outer boundary$r=a$, $N_r=M_r=w^0=0$ for both circular plate and annular plate.

At inner boundary $r=b$,
$u^0=\alpha_1^0=\mathrm{d}w^0/\mathrm{d}r=0$ for circular plates and
$N_r=M_r=w^0=0$ for annular plates.

The Nth-degree Chebyshev polynomial T_N^* has N zeroes at

$$\xi_i=\frac{1}{2}\left\{1+\cos\left[\frac{(2*i-1)\pi}{2N}\right]\right\}\quad i=1,2,\ldots,N. \tag{20}$$

By forcing the satisfaction of each of the three differential equations at the $(N-1)$ zeroes of $T^*_{(N-1)}(\xi)$, $0\le\xi\le1$—the $(N-1)$th degree shifted Chebyshev Polynomial, along with the stipulation of the three boundary conditions at each edge the dynamic equilibrium equations, can be expressed by a set of algebraic equations as

$$\begin{aligned}
&[L_{11}]\{U\}+[L_{12}]\{\alpha\}+[L_{13}]\{q\}+[L_{14}]\{W\}=\{0\}\\
&[L_{21}]\{U\}+[L_{22}]\{\alpha\}+[L_{23}]\{q\}+[L_{24}]\{W\}=\{0\}\\
&[L_{31}]\{U\}+[L_{32}]\{\alpha\}+[L_{33}]\{q\}+[L_{34}]\{W\}\\
&\quad=-\lambda^2[M]\{W\}
\end{aligned} \tag{21}$$

where $\{U\}$, $\{\alpha\}$, $\{q\}$ and $\{W\}$ are the vectors containing the unknown coefficients which are defined by following equations

$\{U\}^T=\{U_1,U_2,\ldots,U_{n+1}\}$,

Table 1 Free vibration frequencies of isotropic circular plate for convergence study

$\upsilon=0.3$, boundary condition: clamped				
Number of terms	Frequency parameter $\lambda=\omega a^2\sqrt{\rho h/D}$			
	1	2	3	4
8	10.007	37.518	80.288	141.926
10	10.007	37.514	79.617	132.955
12	10.007	37.514	79.617	132.568

Table 2 Fundamental natural frequencies of clamped isotropic circular plates

$\lambda = \omega a^2 \sqrt{\rho h/D}$, $\upsilon = 0.3$

h/*a*	Present (FSDT)	Lin and Tseng (1998) (FSDT)	Liew et al. (1997) (FSDT)	Liew and Yang (1999) (3D)
0.05	10.153	10.047	10.145	–
0.10	10.007	9.812	9.941	9.991
0.15	9.764	9.453	9.628	–
0.20	9.450	9.016	9.240	9.322
0.3	8.703	–	–	8.467
0.4	7.905	–	–	7.600
0.5	7.124	–	–	6.807

Table 3 Fundamental natural frequencies of clamped–clamped isotropic annular plate

$\lambda = \omega a^2 \sqrt{\rho h / D}$, $\upsilon = 0.3$

b/*a*	*a*/*h*	Present (FSDT)	Lin and Tseng (1998) (FSDT)	Han and Liew (1999) (FSDT)
0.1	10	25.116	24.480	24.629
	40	27.135	27.466	–
	100	27.241	27.585	–
0.3	10	40.455	38.621	39.398
	40	44.982	45.143	–
	100	45.127	45.406	–
0.5	10	73.192	67.934	70.277
	40	87.931	88.069	–
	100	89.028	89.034	–

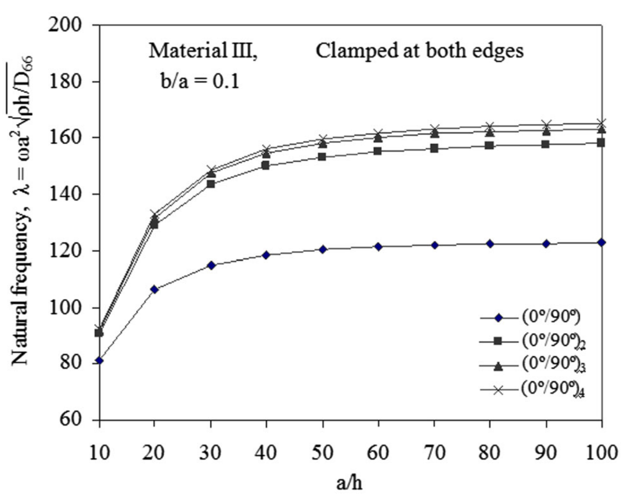

Fig. 2 Effect of number of layers on fundamental natural frequencies of laminated polar orthotropic annular plates

$$\{\alpha\}^T = \{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\},$$
$$\{q\}^T = \{W_1, W_2\},$$
$$\{W\}^T = \{W_3, W_4, \ldots, W_{n+1}\}. \tag{22}$$

The Eq. (21) can be written together in matrix form as

$$\begin{bmatrix} [\bar{L}_{11}] & [\bar{L}_{12}] \\ [\bar{L}_{21}] & [\bar{L}_{22}] \end{bmatrix} \begin{Bmatrix} \{x\} \\ \{y\} \end{Bmatrix} = \lambda^2 \begin{bmatrix} 0 & 0 \\ [\bar{M}_{21}] & [\bar{M}_{22}] \end{bmatrix} \begin{Bmatrix} \{x\} \\ \{y\} \end{Bmatrix} \tag{23}$$

where $\{x\}^T = \{\{U\}^T\{\alpha\}^T\{q\}^T\},\ \{y\}^T = \{W\}^T.$ (24)

By matrix condensation, Eq. (23) can be rewritten as

$$[\bar{K}]\{y\} = \lambda^2 [\bar{M}]\{y\}. \tag{25}$$

The solution of the above eigenvalue problem leads to the determination of the natural frequencies and mode shapes of the laminated orthotropic circular and annular plates undergoing axisymmetric vibrations.

Results and discussion

A C-program developed by Antia (2002) is used in the present work for the free vibration analysis of laminated polar orthotropic circular and annular plates based on the solution methodology described in the preceding sections. Convergence and comparison studies were made to establish the validity of the method. Results of parametric

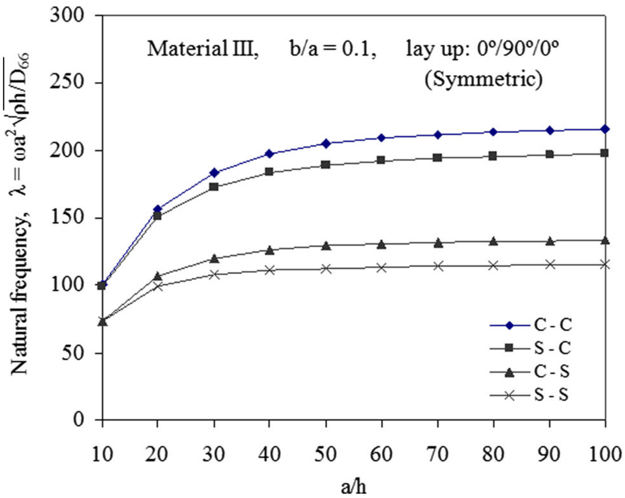

Fig. 3 Effect of boundary conditions on fundamental natural frequencies of laminated polar orthotopic annular plates

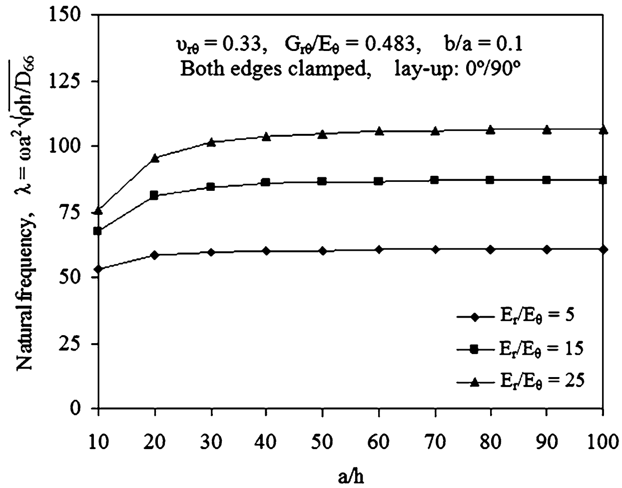

Fig. 5 Effect of orthotropy ratio on fundamental natural frequencies of laminated polar orthotropic annular plates

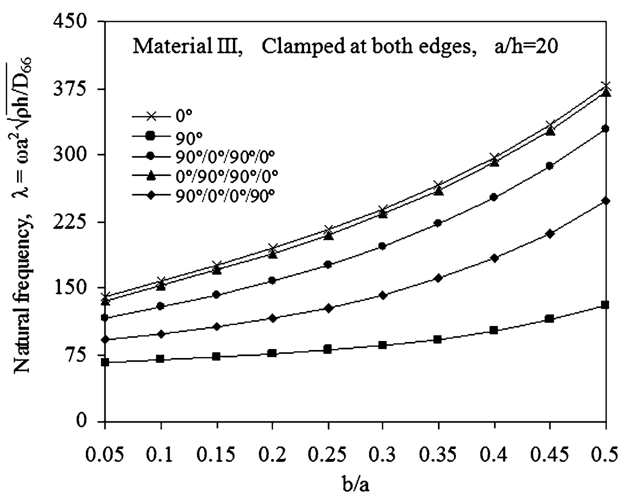

Fig. 4 Effect of hole radius on fundamental natural frequencies of laminated polar orthotropic annular plates

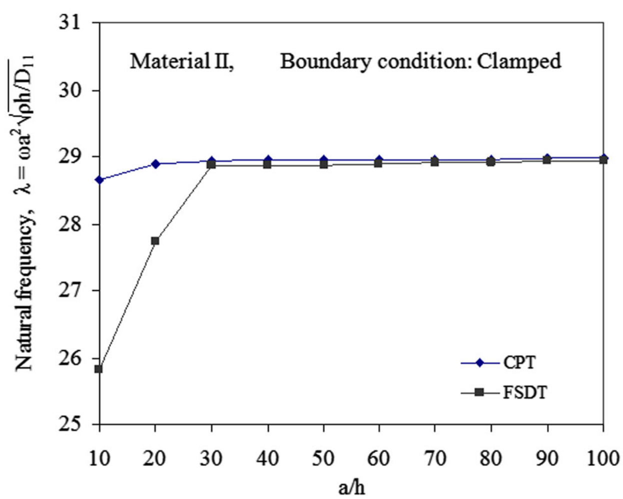

Fig. 6 Effect of shear deformation on fundamental natural frequencies of circular plates

studies are also presented. In all the results presented here, the orientations of the fibers in the layers are identified either as 0° or 90°, depending upon whether the layer is reinforced radially or circumferentially. The material properties along the principal directions are assumed to be the same in all the layers.

Free vibration frequencies of clamped isotropic circular plate are shown in Table 1. It can be seen from the results that converged results are obtained with 10–12 terms of the Chebyshev series approximation. Comparisons between the present results and those of the existing results based on the classical plate theory, three-dimensional plate theory and first-order shear deformation theory are made. Table 2 shows the fundamental natural frequencies of clamped isotropic circular plate which are in good agreement with the results obtained by Lin and Tseng (1998).

Good agreement between the present results and those of Lin and Tseng (1998) and Han and Liew (1999) for isotropic annular plates clamped at both edges is also seen in Table 3

Parametric study

Free vibration analysis has been carried out on orthotropic circular and annular plates. Material properties of specimens used in this study as obtained from literature (Lin and Tseng 1998) are as given below.

Material I: $E_\theta/E_r = 5$, $G_{r\theta}/E_r = 0.35$, $G_{rz}/E_r = 0.292$, $G_{\theta z}/E_r = 0.292$, $\upsilon_{\theta r} = 0.3$, $\rho = 1.0$.

Table 4 Fundamental frequencies of polar orthotropic laminated circular plates composed of different materials; $\lambda = \omega a^2 \sqrt{\rho h / D_{11}}$

BC	a/h	III	II	III/II/III/II	III/II/II/III	II/III/III/II
Clamped	10	26.248	13.404	15.226	21.132	13.456
	20	27.735	13.867	15.869	21.789	14.189
	50	28.867	14.003	16.071	21.985	14.381
	100	28.940	14.031	16.100	22.022	14.414
Simply supported	10	21.644	8.392	10.522	15.910	8.689
	20	22.690	8.512	10.714	16.182	8.873
	50	23.011	8.542	10.789	16.261	8.936
	100	23.058	8.548	10.798	16.268	8.944

Table 5 Fundamental frequencies of polar orthotropic laminated annular plates composed of different materials; $\lambda = \omega a^2 \sqrt{\rho h / D_{11}}$

BC	b/a	a/h	III	II	III/II/III/II	III/II/II/III	II/III/III/II
C–C	0.1	10	46.409	28.146	30.287	41.634	26.921
		20	55.849	31.223	34.113	45.553	31.220
		50	60.127	32.300	35.464	46.937	32.864
		100	60.712	32.453	35.678	47.125	33.111
	0.5	10	91.516	71.796	71.319	89.052	65.952
		20	105.339	85.002	83.017	96.991	82.563
		50	110.485	90.203	87.437	99.602	89.984
		100	111.282	91.021	88.147	100.015	91.211
S^{in}–C^{out}	0.1	10	47.388	26.034	28.387	47.629	25.271
		20	55.840	28.130	30.575	47.896	28.867
		50	57.619	28.867	31.037	48.041	29.196
		100	63.231	28.915	31.376	48.337	29.482
	0.5	10	83.800	62.683	62.907	78.015	58.752
		20	93.823	71.031	70.499	83.131	69.741
		50	97.312	74.023	73.166	84.758	74.084
		100	97.823	74.473	73.561	84.972	74.785
C^{in}–S^{out}	0.1	10	39.467	20.471	23.219	33.171	19.972
		20	46.359	22.193	25.612	35.860	22.443
		50	49.183	22.777	26.477	36.788	23.370
		100	49.727	22.877	26.594	36.948	23.515
	0.5	10	73.741	53.467	54.137	67.040	50.539
		20	81.649	59.665	59.911	70.941	58.813
		50	84.394	61.839	61.898	72.168	61.981
		100	84.818	62.160	62.197	72.357	62.475
S–S	0.1	10	40.403	18.575	21.320	36.054	18.432
		20	46.092	19.561	22.055	38.258	19.868
		50	48.235	19.764	22.062	37.456	20.372
		100	49.758	19.765	22.205	37.739	20.338
	0.5	10	66.860	44.220	46.299	57.241	42.969
		20	72.074	46.978	49.242	59.328	46.906
		50	73.801	47.847	50.175	59.954	48.213
		100	74.043	47.973	50.315	60.051	48.410

Material II: $E_\theta/E_r = 50$, $G_{r\theta}/E_r = 0.6613$, $G_{rz}/E_r = 0.5511$, $G_{\theta z}/E_r = 0.5511$, $\upsilon_{\theta r} = 0.26$, $\rho = 1.0$.

Material III (ultra-high-modulus graphite epoxy): $E_r = 310 \times 10^3$ N/mm^2, $E_\theta = 6.2 \times 10^3$ N/mm^2, $G_{r\theta} = 4.1 \times 10^3$ N/mm^2, $\upsilon_{r\theta} = 0.26$, $\rho = 1.613 \times 10^3$ kg/m^3.

In all the parametric studies reported herein, a 12-term solution is adopted hereafter, in computation of the free vibration response of different plates.

The results of parametric study to know the effect of the number of layers on the free vibration frequencies of a

Table 6 Fundamental natural frequencies of polar orthotropic laminated circular plates: effect of fiber orientation; material III; $\lambda = \omega a^2 \sqrt{\rho h / D_{66}}$

BC	a/h	0°	90°	90°/0°/90°/0°	0°/90°/90°/0°	90°/0°/0°/90°
Clamped	10	48.552	32.793	49.549	45.539	41.959
	20	56.622	34.868	57.553	51.420	46.584
	50	64.415	35.522	61.475	52.646	48.173
	100	71.887	35.619	67.681	60.335	48.416
Simply supported	10	26.726	6.482	25.819	10.867	27.735
	20	27.735	9.950	26.726	12.598	28.867
	50	28.329	19.611	26.547	21.320	29.437
	100	28.867	22.880	26.130	28.860	30.151

Table 7 Fundamental natural frequencies of polar orthotropic laminated annular plates: effect of fiber orientation; material III; $\lambda = \omega a^2 \sqrt{\rho h / D_{66}}$

BC	b/a	a/h	0°	90°	90°/0°/90°/0°	0°/90°/90°/0°	90°/0°/0°/90°
C–C	0.1	10	100.321	59.215	90.759	98.966	76.628
		20	157.582	69.404	128.809	152.979	97.916
		50	207.345	73.185	153.266	197.449	108.503
		100	219.159	74.597	158.034	207.613	110.397
	0.5	10	208.288	115.794	198.968	207.256	175.899
		20	377.587	130.811	329.368	371.544	247.840
		50	629.316	136.171	459.363	602.475	290.864
		100	727.778	136.999	493.563	686.964	298.940
S^{in}–C^{out}	0.1	10	99.800	59.496	89.550	98.342	74.680
		20	152.127	69.673	122.840	147.313	92.489
		50	191.530	74.247	141.676	182.331	100.316
		100	200.080	75.701	145.156	189.694	101.529
	0.5	10	203.615	105.514	191.741	202.154	163.539
		20	355.177	116.247	301.101	347.776	216.777
		50	544.734	119.883	391.840	519.034	243.975
		100	606.004	120.429	412.638	571.102	248.759
C^{in}–S^{out}	0.1	10	74.124	49.962	67.450	73.107	58.035
		20	107.670	57.792	89.694	104.313	70.728
		50	130.732	61.233	98.010	124.902	76.673
		100	135.444	61.244	104.507	129.024	77.732
	0.5	10	184.427	92.648	170.325	182.361	141.251
		20	307.002	101.113	255.906	299.610	181.071
		50	449.466	103.975	321.611	427.412	200.401
		100	491.889	104.404	335.842	463.340	203.700
S–S	0.1	10	73.821	50.144	66.328	72.490	55.971
		20	100.055	58.261	83.888	96.673	65.938
		50	114.123	61.061	92.253	109.304	69.572
		100	116.713	61.221	93.658	111.601	69.843
	0.5	10	184.302	83.565	166.689	181.608	127.837
		20	284.440	89.122	229.536	274.617	150.414
		50	362.666	90.871	265.092	342.997	159.272
		100	379.978	91.135	271.662	357.530	160.664

laminated annular plate with both the edges clamped are presented in Fig. 2. Figure 3 presents the results of a study conducted to study the effect of boundary conditions on free vibration frequencies of a laminated polar orthotropic annular plate. Effect of the size of the hole on the fundamental frequency of annular plates clamped at both edges is shown in Fig. 4.

Figure 5 shows the results of a study conducted to know the effect of orthotropy ratio on the free vibration frequencies of a two-layered asymmetric cross-ply annular plate with both the edges clamped.

A comparison of the natural frequencies calculated from the present shear deformation theory with those predicted by CPT is presented in Fig. 6. It can be observed that the effect of shear deformation is to decrease the free vibration frequencies in case of thick plates.

Fundamental natural frequencies of polar orthotropic laminated circular plates with clamped and simply supported boundary conditions are listed in Table 4. Results show that natural frequencies are influenced by stacking sequence and the order of the magnitude of the fundamental frequency for the five different laminates. For annular plates, the effects of stacking sequence on natural frequencies when the inner and outer edges are either clamped or simply supported are illustrated in Table 5 and are similar to those for circular plates.

Results of the fundamental frequency of several polar orthotropic laminated circular plates with clamped or simply supported boundary conditions are shown in Table 6. The ultra-high-modulus graphite epoxy composites are used in the examples. They reveal that among these different stacking sequences, the smallest natural frequency occurs when the plate is composed of laminae in which fibers are oriented in circumferential direction only. It seems to be reasonable, since the displacement and curvature of the first vibration mode of the plates are varied in the radial direction only. Hence, the laminated plate having higher stiffness in the radial direction would produce higher natural frequency and vice versa. Because the fibers are placed along circumferential direction in this laminate, the stiffness in the radial direction is smaller than any other laminates.

Fundamental frequencies of C–C, S^a–C, C^a–S and S–S polar orthotropic laminated annular plates listed in Table 7 show that the order of the magnitude of the fundamental frequency for these five laminates is (0°) > (0°/90°//90°/0°) > (90°/0°/90°/0°) > (90°/0°/0°/90°) > (90°). The same behavior has been found for laminated plates given by Lin and Tseng (1998). Typical axisymmetric mode shapes (torsionless) corresponding to fundamental natural frequencies for the laminated circular and annular plates with different boundary conditions, showing the effect of stacking sequences, are plotted in Figs. 7, 8, 9, 10, 11 and 12.

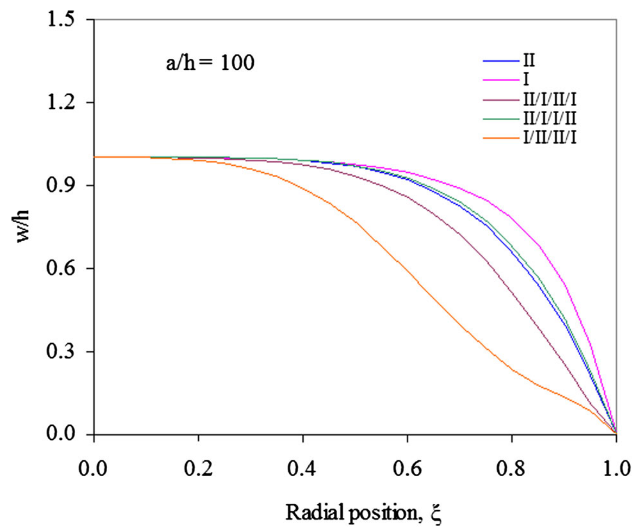

Fig. 8 Fundamental mode shapes for laminated polar orthotropic simply supported circular plates

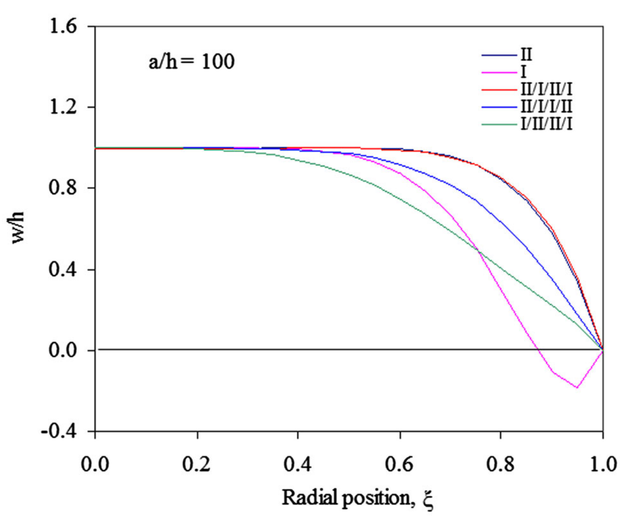

Fig. 7 Fundamental mode shapes for laminated polar orthotropic clamped circular plates

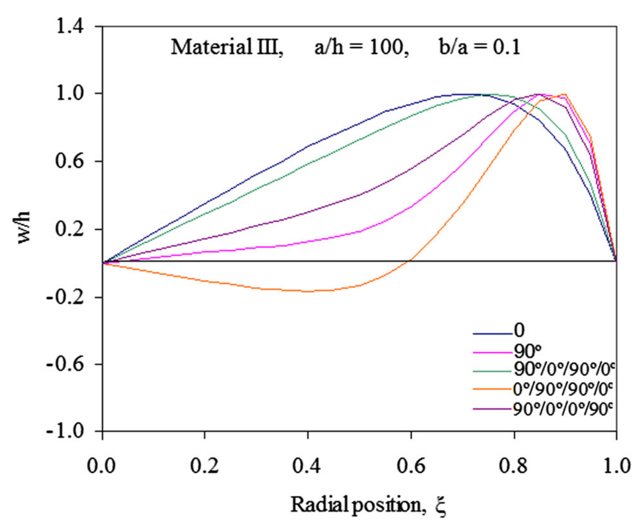

Fig. 9 Fundamental mode shapes for laminated polar orthotropic C–C annular plates

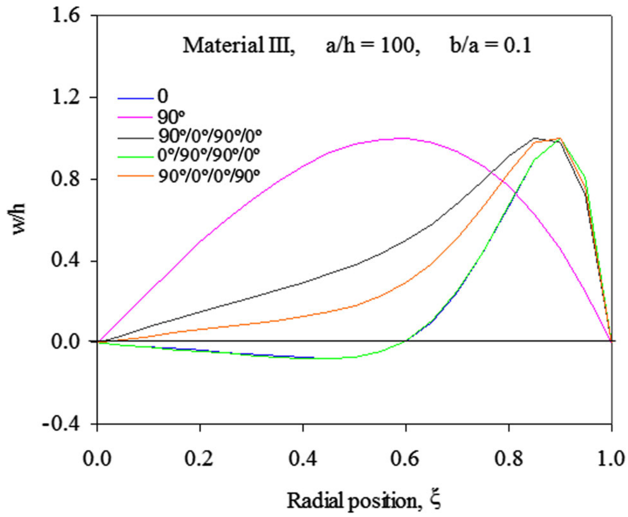

Fig. 10 Fundamental mode shapes for laminated polar orthotropic S^a–C annular plates

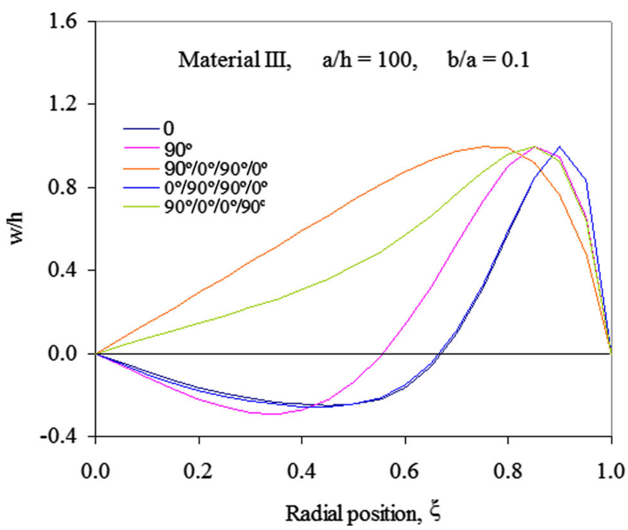

Fig. 11 Fundamental mode shapes for laminated polar orthotropic C^a–S annular Plates

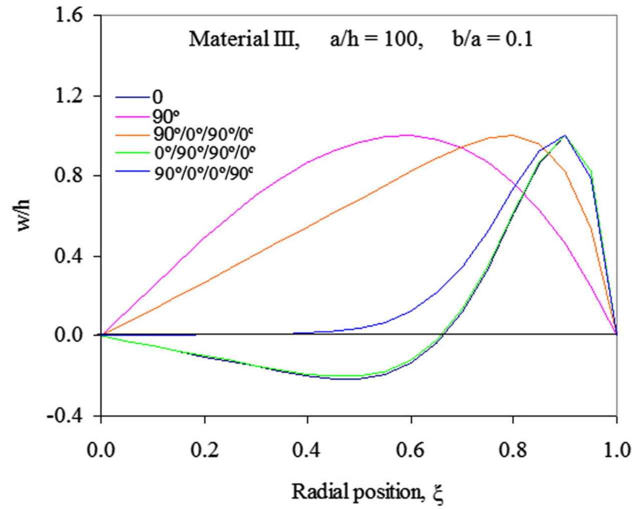

Fig. 12 Fundamental mode shapes for laminated polar orthotropic S–*S* annular plates

Conclusions

Free vibration characteristics of composite circular and annular plates were studied in detail, with formulation based on a first-order shear deformation theory and a solution methodology employing the Chebyshev collocation technique. Convergence tests were conducted for the Chebyshev collocation technique and it can be seen that there is excellent convergence even when we take four or six terms in the series for the problem considered. Further, numerical results have aided to conclude that

- The solution method, based on collocating the equations of motion at Chebyshev zeroes as proposed herein, developed systematically in polar co-ordinates, is reliable and effective for finding natural frequencies and mode shapes of polar orthotropic circular and annular plates. The fundamental frequency of polar orthotropic laminated annular plates increases with an increase in number of layers, hole size and orthotropy ratio. Fundamental frequencies are higher for clamped boundary conditions.
- Transverse shear effects are more significant for polar orthotropic laminated plates than isotropic plates. Further, the transverse shear effects are negligible in case of thin plates.
- For polar orthotropic laminated circular and annular plates with clamped edges, the laminate stacked with all layers having fibers oriented along the radial direction has the highest fundamental frequency.
- Parametric studies conclude that free vibration frequencies are dependent not only on the radius to thickness ratio, but also on plate parameters such as the fiber orientation, lamination sequence, hole diameter and the boundary conditions.

Author contributions A. Powmya carried out the study under the guidance of MCN. MCN also participated in the sequence alignment and drafted the manuscript. Both authors read and approved the final manuscript.

Conflict of interest The authors declare that they have no competing interests.

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